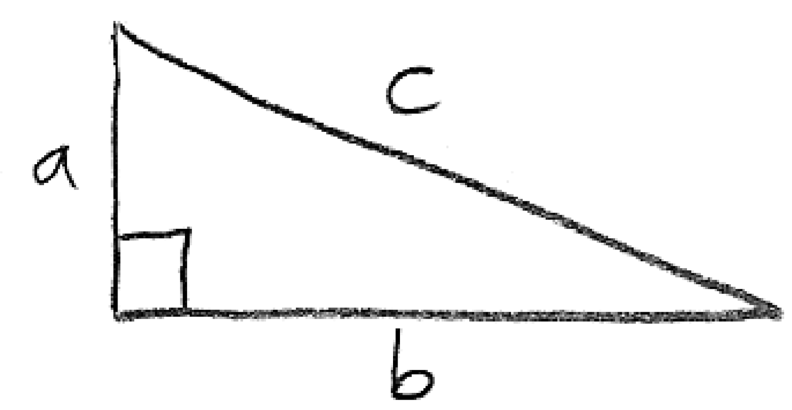
Right Triangle Problem from American Invitational Mathematics Exam (AIME) 1991(?)

Recalled from my high school days, I was unable to solve this problem during the actual contest. I found one solution shortly after, and just recently wondered how many different ways I could find to solve it (preferably utilizing/emphasizing different branches of mathematics).

The problem, paraphrased from memory:

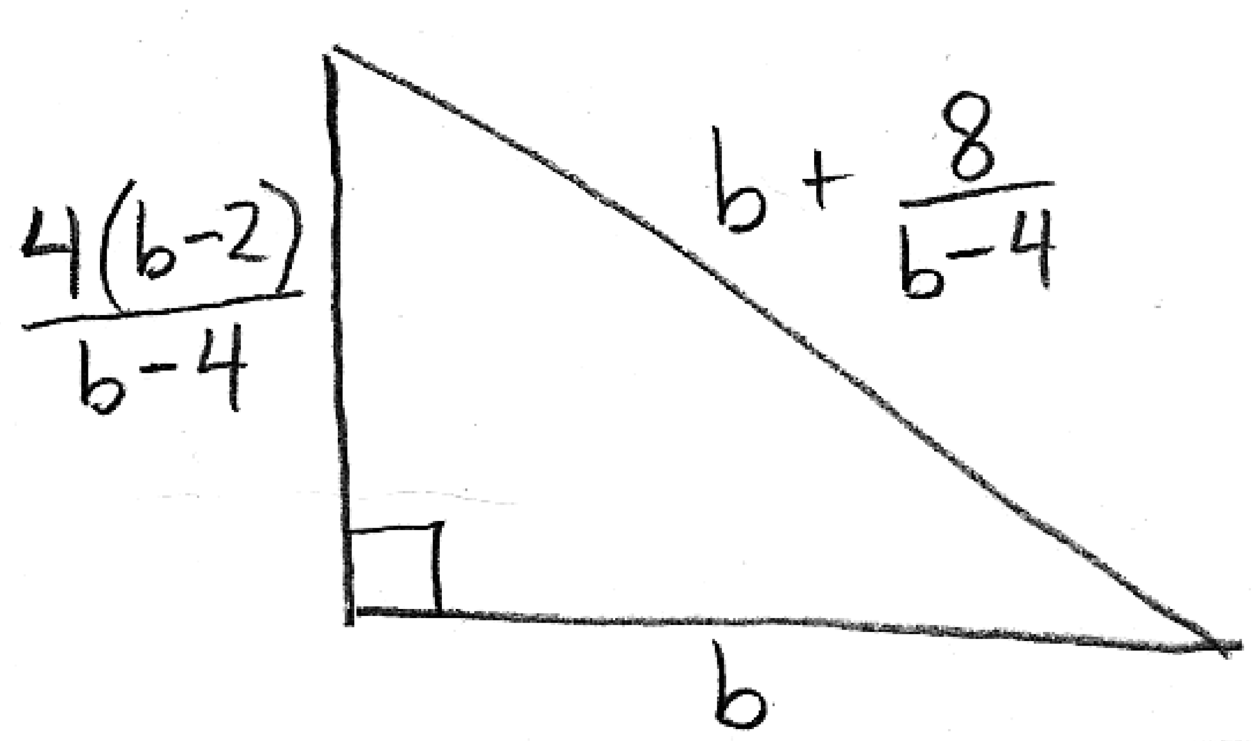
> How many right triangles exist such that the area in square units is equal to the perimeter in linear units. Prove your answer.

Solution #1: Algebraic



$$\text{Perimeter} = \text{Area}$$
$$a+b+c = \tfrac{1}{2}ab$$
$$a+b+\sqrt{a^2+b^2} = \tfrac{1}{2}ab$$
$$2\sqrt{a^2+b^2} = ab-2a-2b$$
$$4(a^2+b^2) = a^2b^2+4a^2+4b^2-4a^2b-4ab^2+8ab$$
$$0 = a^2b^2-4a^2b-4ab^2+8ab$$
$$0 = ab-4a-4b+8$$
$$ab-4a = 4b-8$$
$$a(b-4) = 4(b-2)$$
$$a = \frac{4(b-2)}{b-4}$$

$$c = \sqrt{a^2+b^2} = \sqrt{\frac{16(b-2)^2}{(b-4)^2}+b^2} = \frac{\sqrt{16(b-2)^2+b^2(b-4)^2}}{b-4} = \frac{\sqrt{b^4-8b^3+32b^2-64b+64}}{b-4}$$
$$= \frac{b^2-4b+8}{b-4} = b+\frac{8}{b-4}$$



For this triangle, Perimeter[units] = Area[units^2] for all real $b>4$

($b \geq 4+2\sqrt{2}$ gives all unique solutions for which $b \geq a$)

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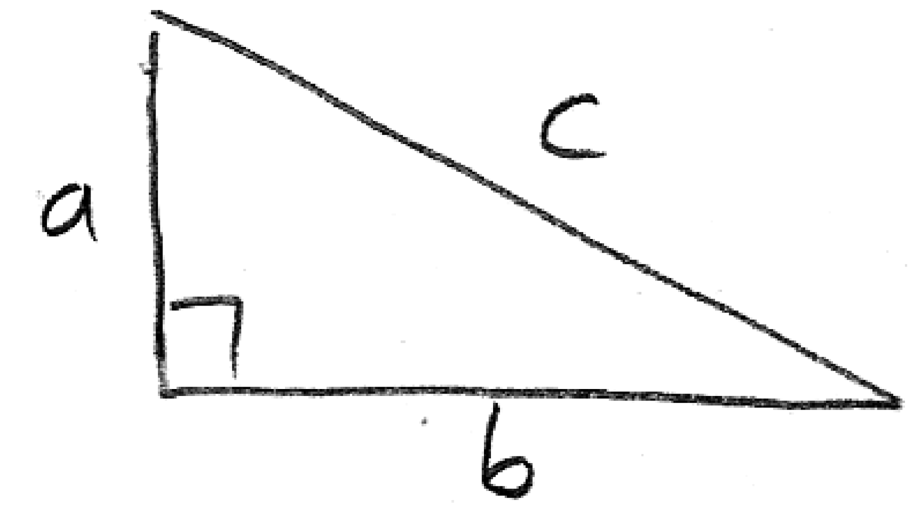
Solution #2: Geometric/Dimensional Analysis

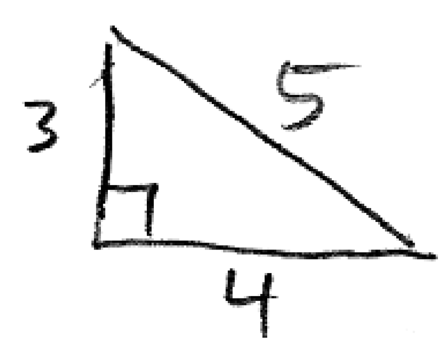


For any given right triangle with legs of length a & b and hypotenuse of length c, the ratio r of perimeter [units] to area [units^2] is

$$r = \frac{\text{Perimeter}}{\text{Area}} \frac{P}{A} = \frac{a+b+c}{\frac{1}{2}ab} = \frac{2\left(a+b+\sqrt{a^2+b^2}\right)}{ab}$$

Also: $P = rA$

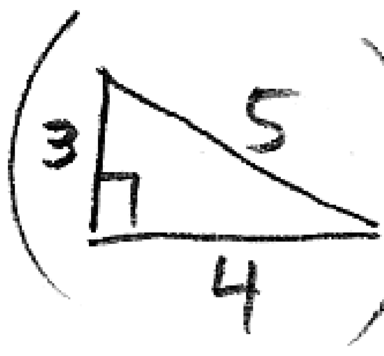
For example:



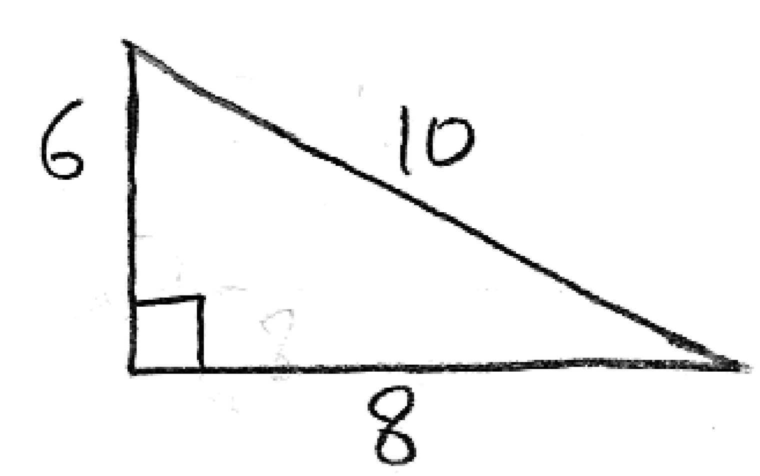
$$\Rightarrow r = \frac{3+4+5}{\frac{1}{2}(3)(4)} = \frac{12}{6} = 2$$

Multiplying all sides by r, the perimeter is increased $(r>1)$ or decreased $(r<1)$ by a factor of r while the area is increased or decreased by a factor of r^2.
Therefore, the new perimeter is rP and the new area is r^2A.

The new ratio of perimeter to area is $\frac{rP}{r^2A} = \frac{r(rA)}{r^2A} = \frac{r^2A}{r^2A} = 1$,
and therefore the perimeter [units] = area [units^2]

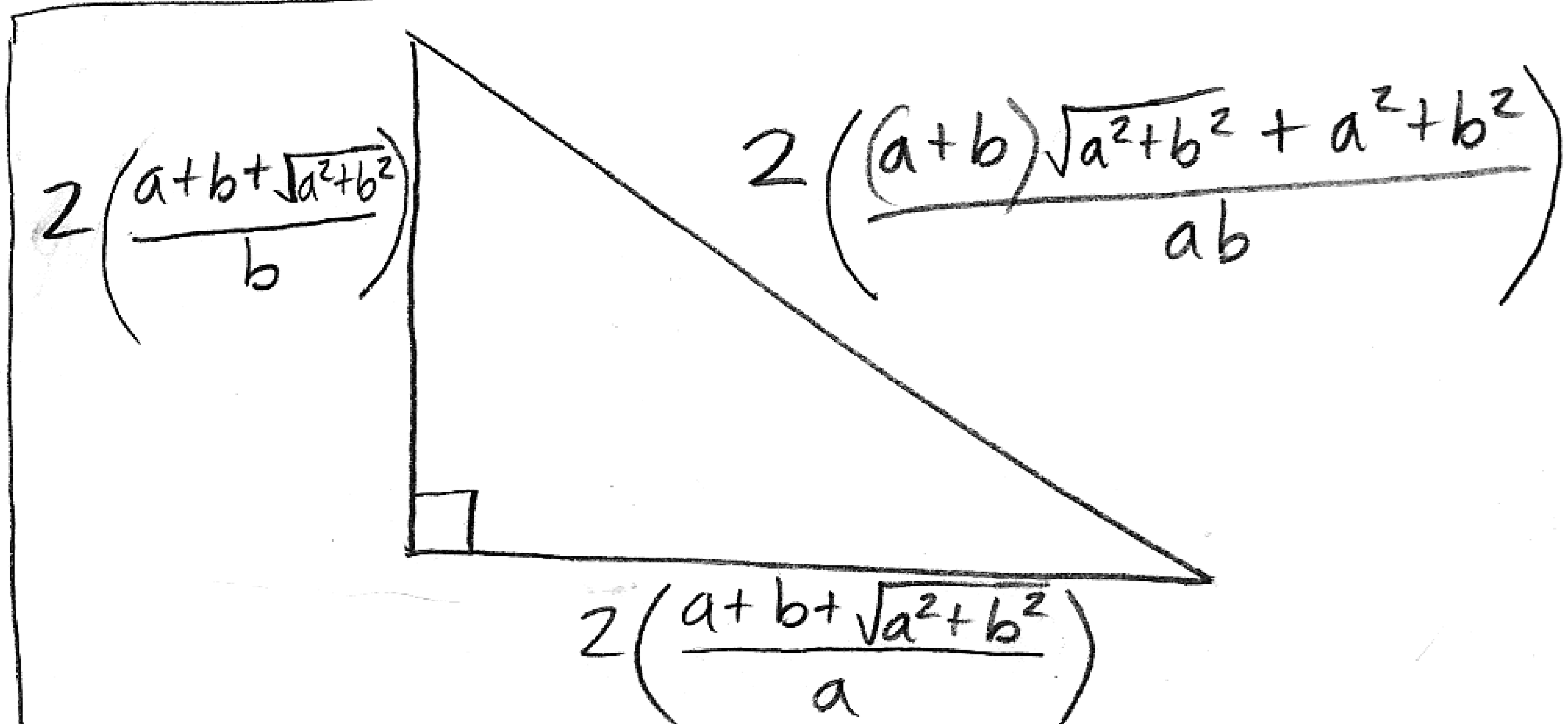
For example: (

) $\cdot 2 \Rightarrow$



$P = 6+8+10 = 24$ ✓
$A = \frac{1}{2}(6)(8) = 24$

Therefore, multiplying the original a-b-c triangle by r gives



For all real $a>0$ and all real $b>0$,
Perimeter [units] = Area [units^2]
For this triangle.

(Some negative values of a and/or b also yield non-unique solutions)

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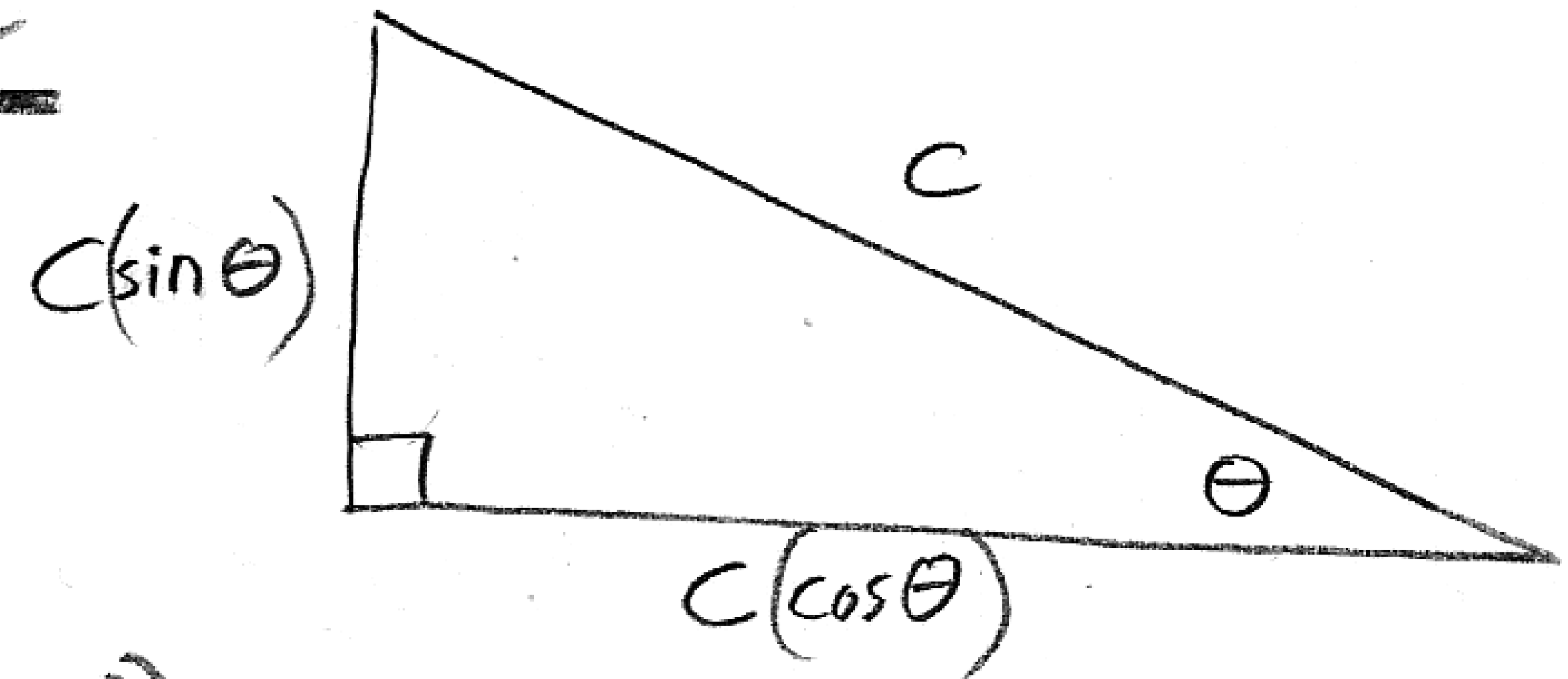
Solution #3: Trigonometric

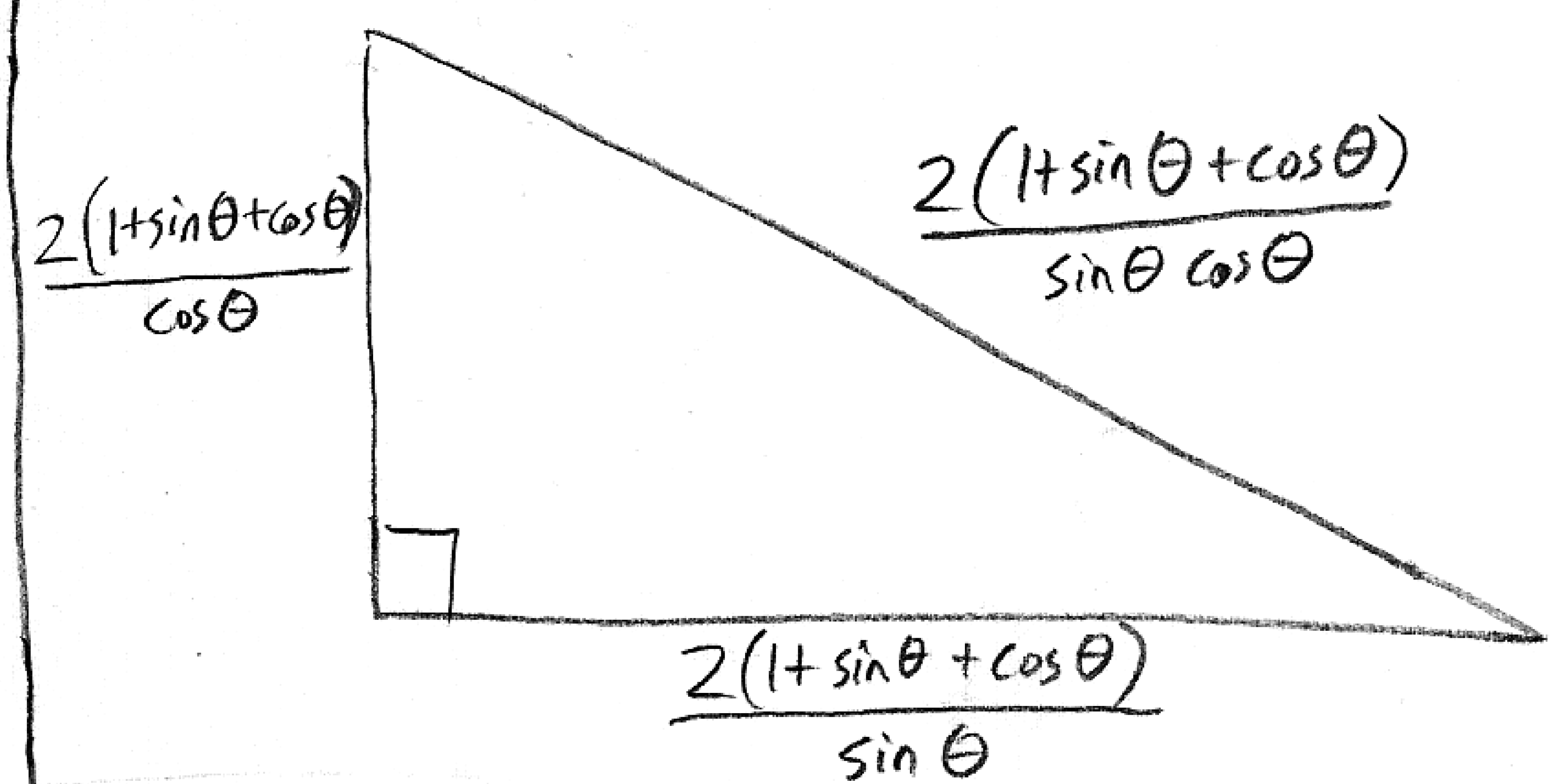


Perimeter = Area

$$c\sin\theta + c\cos\theta + c = \frac{1}{2}\left(c(\sin\theta)\right)\left(c(\cos\theta)\right)$$

$$2(\sin\theta + \cos\theta + 1) = c\sin\theta\cos\theta$$

$$c = \frac{2(1+\sin\theta+\cos\theta)}{\sin\theta\cos\theta}$$



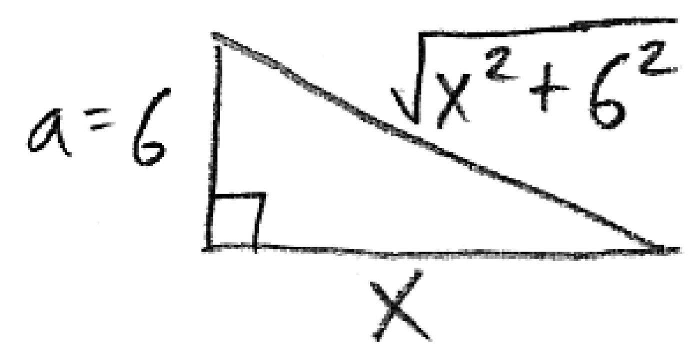
For this triangle,
Perimeter[units] = Area [units^2]
for all real $0<\theta$[radians]$<\frac{\pi}{2}$

($0<\theta\le\frac{\pi}{4}$ gives all unique solutions)

Solution #4a: Graphical/Technological

Pick a positive real number a to represent the leg of a right triangle (other solutions show that $a>4$ is necessary to yield real solutions)

Consider $a=6$ for example:

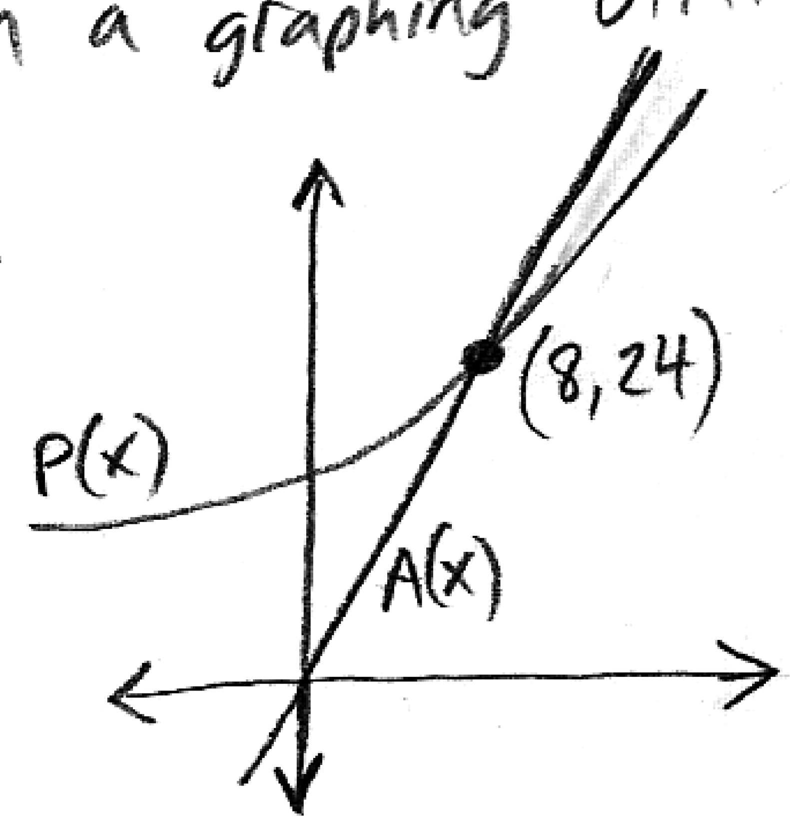


Area $A(x) = \frac{x\cdot 6}{2} = 3x$

Perimeter $P(x) = 6+x+\sqrt{x^2+6^2}$

Graph both $A(x)$ and $P(x)$ on a graphing utility and find intersection

For example of $a=6$:



At point of intersection:

x coordinate gives length of other leg.

y coordinate gives the resulting numerical value of the area and perimeter

Solution #4b: Numerical/Technological

Same as #4a, except examine a table of values for $A(x)$ and $P(x)$ and find the value of x such that $A(x) = P(x)$

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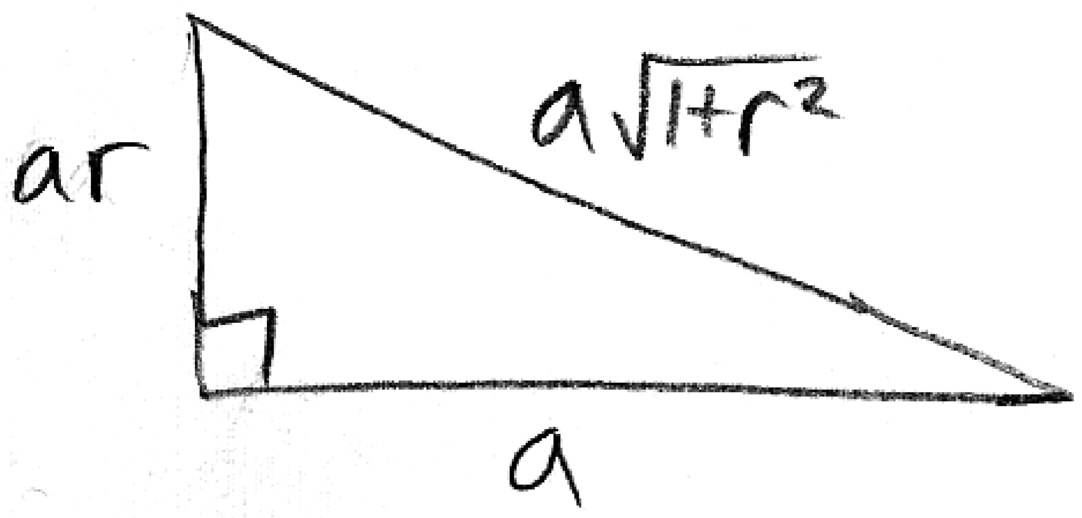
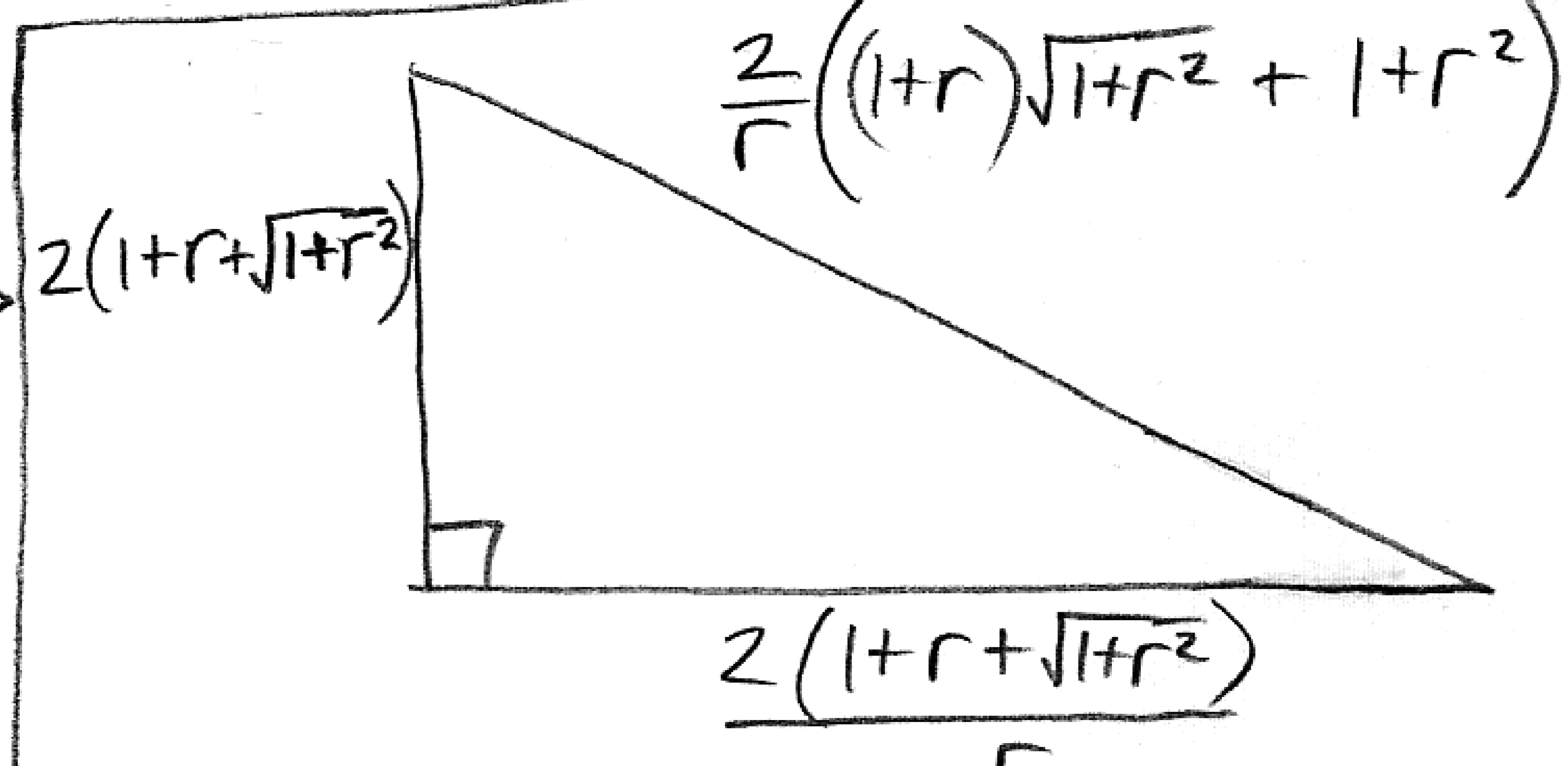
Solution #5: Algebraic (again!)

For a right triangle with one leg of length a, let r equal the ratio of the lengths of the legs



$$\text{Perimeter} = \text{Area}$$
$$a + ar + a\sqrt{1+r^2} = \tfrac{1}{2}(a)(ar)$$
$$a\left(1+r+\sqrt{1+r^2}\right) = \tfrac{1}{2}a^2 r$$
$$a = \frac{2\left(1+r+\sqrt{1+r^2}\right)}{r}$$

Note similarity to Solution #2 result

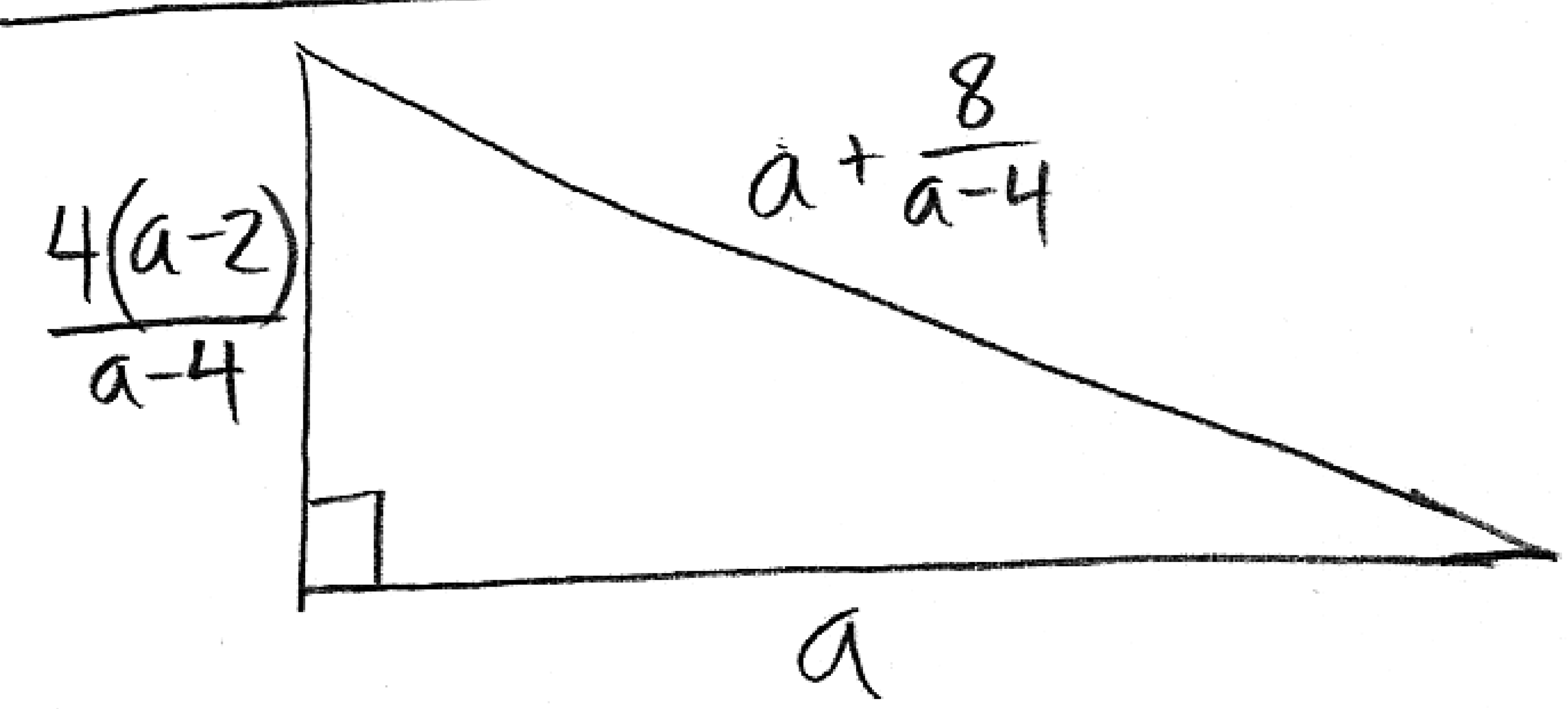


For this triangle,
perimeter [units] = Area [units²]
for all real $r > 0$
($r \geq 1$ yields all unique solutions for which $ar \geq a$)

or

Solve for r in terms of a:

$$ar = 2 + 2r + 2\sqrt{1+r^2}$$
$$(a-2)r - 2 = 2\sqrt{1+r^2}$$
$$(a-2)^2 r^2 - 4(a-2)r + 4 = 4 + 4r^2$$
$$(a-2)^2 r - 4(a-2) = 4r$$
$$(a-2)^2 r - 4r = 4(a-2)$$
$$r = \frac{4(a-2)}{(a-2)^2 - 4}$$
$$r = \frac{4(a-2)}{a^2 - 4a}$$
$$r = \frac{4}{a}\left(\frac{a-2}{a-4}\right)$$



For this triangle,
perimeter [units] = Area [units²]
for all real $a > 4$
($a \geq 4 + 2\sqrt{2}$ gives all unique solutions for which $a \geq ar$)

Note similarity to solution #1 result, from which expression for hypotenuse was obtained

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